

# CET: A New Complex Evidence Theory

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## Abstract

Dempster-Shafer evidence theory, as an extension of Probability theory, is widely used in the field of information fusion due to it satisfies weaker conditions than probability theory in dealing with uncertain information. Nevertheless, the description space of the current evidence theory is only a real space, and it cannot effectively describe and process the uncertain information in the face of multidimensional characteristic data and periodic data with phase angle changes. Based on this gap, in this paper, Dempster-Shafer evidence theory is extended to the complex Dempster-Shafer evidence theory. In complex Dempster-Shafer evidence theory, mass function that used to describe the uncertain information extends from the real space to the complex space, named as complex mass function, and the modulus of the mass function indicates the degree of support for the proposition. On this basis, other basic concepts used to describe uncertainty information are also defined and discussed, such as complex belief function, complex plausibility function, etc. In order to perfect the complex Dempster-Shafer evidence theory, the complex Dempster combination rule (CDCR) is supplemented. CDCR is an extension of Dempster combination rule (CDR), which satisfies the commutative and associative laws just as CDR does, and it can degenerate into CDR under certain condition. In addition, we propose a method to generate complex mass function and apply it to target recognition.

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The recognized results show that compared with the mass function of the real plane, the target recognition rate can be larger by using complex mass function to describe the uncertain information.

*Keywords:* Dempster-Shafer evidence theory, complex Dempster-Shafer evidence theory, complex mass function, complex Dempster combination rule

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## 1. Introduction

In nature, human cognition of things is often expressed through uncertain or imprecise information. In order to make this kind of cognitive expression more reasonable, scholars have developed many theories to describe and deal with uncertain or imprecise information, such as Probability theory[1], Dempster-Shafer evidence theory[2, 3], Fuzzy set theory[4], Atanassov's fuzzy set[5], entropy-based[6, 7] and etc[8]. On the one hand, scholars extend these theories to related theories, like evidence reasoning[9], fuzzy soft set[10], information volume[11], interval Atanassov's fuzzy set[12], Pythagorean fuzzy set[13], mass function of quantum[14] and other hybrid methods[15, 16, 17, 18]. On the other hand, since above theories have their own advantages, researchers have applied them to a variety of fields such as target classification[19, 20, 21, 22, 23], cluster analysis[24, 25, 26, 27, 28], medical diagnosis[29, 30, 31, 32, 33], reliability analysis[34, 35, 36, 37, 38], multi-criteria decision making[39, 40, 41, 42, 43] and other cross-application[44, 45, 46, 47, 48].

Dempster-Shafer evidence theory as a mathematical tool for dealing with uncertain information, which is a generalization of Probability theory. Compared with Probability theory, evidence theory satisfies weaker uncertainty conditions, and the multi-element expression of propositions shows that it has sufficient fault-tolerant ability. Dempster combination rule (DCR) can gradually improve the degree of support for the objective proposition and reduce the degree of support for the non-objective proposition. From the perspective of algebraic operations, DCR satisfies the commutative and associative laws of multiplication, which guarantees the stability and symmetry of algebraic operations. Due

25 to these advantages, many decision theories have been extended and applied to various fields around the evidence theory. Whereas, it is found that the description of the uncertain information in the relevant theories of evidence theory is in the real number space, which leads to the lack of effective mathematical tools to describe and process the multidimensional characteristics of the uncertain information. Furthermore, when the uncertain information is presented as periodic data of phase angle change, the existing description method cannot capture this change[49]. Consequently, the current research method has a gap that cannot be ignored when describing and processing uncertain information.

Hence, in this paper, we extend the evidence theory to the complex evidence theory. Specifically including the following contents, the description space of uncertain information expands from real space to complex space. Then, a new mass function is defined in the complex number plane, named as the complex mass function, and the support degree of the complex number mass function to the proposition is non-negative and normative. On this basis, some basic concepts are also mapped from the real space to the complex space. For instance, complex belief function, complex plausibility function, etc. Moreover, a new combination rule is proposed in the complex plane, called complex Dempster combination rule (CDCR). CDCR inherits the advantages of DCR. It can not only gradually improve the support degree of the target proposition, but also satisfy the commutative law and associative law of multiplication, and ensure the stability and symmetry of CDCR in the complex plane. When the complex mass function degenerates to the classical mass function, CDCR degenerates to DCR. In addition, we propose a method to generate complex mass function. In the environment of complex evidence theory, complex mass function is applied to target recognition to improve the accuracy of target recognition.

The remain of the paper is structured as follows. The relevant knowledge of evidence theory is introduced in Section 2. Section 3 proposes the complex Dempster-Shafer evidence theory, and some numerical examples are given to explain it. In Section 4, we propose a method to generate complex mass function and apply it to target recognition. We summarize this work in Section 5.

## 2. Preliminaries

A brief review about Dempster-Shafer evidence theory is introduced in the section[2, 3].

**Definition 1.** (*Frame of discernment*)

60 Supposing there exists a positive definite non-empty set whose elements satisfy mutual exclusion  $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ , then this set is a frame of discernment (FOD), its power set is described as follows:

$$2^\Theta = \{\emptyset, \{\theta_1\}, \{\theta_2\}, \dots, \{\theta_n\}, \{\theta_1, \theta_2\}, \dots, \Theta\} \quad (1)$$

where  $\emptyset$  is called the empty set.

**Definition 2.** (*Mass function*)

65 The mass function is used to indicate the degree of support for the proposition, it is defined as

$$m : 2^\Theta \rightarrow [0, 1] \quad (2)$$

which satisfies the following condition:

$$m(\emptyset) = 0 \text{ and } \sum_{A \subseteq 2^\Theta} m(A) = 1 \quad (3)$$

When  $m(A) > 0$ ,  $A$  is also called a focal element of the mass function,  $m(A)$  indicates the degree of support for proposition  $A$ .  $m(A)$  is also called the basic probability assignment function (BPA).  
70

**Definition 3.** (*Dempster combination rule*)

There are two BPAs  $m_1$  and  $m_2$  in FOD  $\Theta$ , the Dempster combination rule  $m_1 \oplus m_2$  is formulated as follows:

$$m(A) = \begin{cases} \frac{\sum_{i,j:A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{\sum_{i,j:A_i \cap B_j \neq \emptyset} m_1(A_i) m_2(B_j)} & A \neq \emptyset \\ 0 & A = \emptyset \end{cases} \quad (4)$$

where  $K = 1 - \sum_{i,j:A_i \cap B_j \neq \emptyset} m_1(A_i) m_2(B_j)$ ,  $K$  is a normalization constant, called conflict coefficient.  $K \in [0, 1]$ ,  $K = 0$  shows that there is no conflict between  $m_1$  and  $m_2$ , and  $K = 1$  shows that there is complete conflict between  $m_1$  and  $m_2$ . Under normal circumstances,  $0 \leq K < 1$ .  
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### 3. Complex Dempster-Shafer evidence theory

In the section, we will introduce complex Dempster-Shafer evidence theory.

80 There is a frame of discernment  $\mathbb{K} = \{\kappa_1, \kappa_2, \dots, \kappa_n\}$ , the power set of  $\mathbb{K}$  is described as follows:

$$2^{\mathbb{K}} = \{\emptyset, \{\kappa_1\}, \{\kappa_2\}, \dots, \{\kappa_n\}, \{\kappa_1, \kappa_2\}, \dots, \mathbb{K}\} \quad (5)$$

The complex mass function on frame of discernment  $\mathbb{K}$  is defined below:

**Definition 4.** (*Complex mass function*)

$$|\mathbb{CM}| : 2^{\mathbb{K}} \rightarrow [0, 1] \quad (6)$$

85  $|\mathbb{CM}|$  is modulus of complex number  $\mathbb{CM}$ , and satisfies

$$\mathbb{CM}(\emptyset) = 0, \quad (7)$$

$$\mathbb{CM}(A) = \mathbb{M}(A) e^{i\theta_A} = x_A + iy_A \quad (8)$$

$$\sum_{A \subseteq \mathbb{K}} \mathbb{M}(A) = \sum_{A \subseteq \mathbb{K}} \sqrt{x_A^2 + y_A^2} = 1 \quad (9)$$

Obviously, *Eq.10* is the equivalent of Euler's formula, the phase angle  $\theta_A = \arctan \frac{y}{x}$ , and  $\theta_A \in [-\pi, \pi]$ .  $\mathbb{M}(A)$  or  $\sqrt{x_A^2 + y_A^2}$  is represents how strongly  
90 this evidence supports the proposition  $A$ , not  $x_A$ . When  $\theta_A = 0$ , complex mass function is degenerated to traditional mass function,  $\mathbb{CM}(A)$  becomes a real number. The complex mass function is also known as the complex basic probability assignment function (CPBA). Now, a numerical example explains the difference between CPBA and BPA.

95 **Example 3.1** Supposing that there is a CBPA  $m$  in  $\mathbb{K} = \{\kappa_1, \kappa_2, \kappa_3\}$ .

$$\mathbb{CM}(\kappa_2) = 0.4832 + 0.1242i = 0.4989e^{i\arctan \frac{0.1242}{0.4832}},$$

$$\mathbb{CM}(\kappa_2, \kappa_3) = 0.2403 + 0.1529i = 0.2848e^{i\arctan \frac{0.1529}{0.2403}},$$

$$\mathbb{CM}(\kappa_1, \kappa_2, \kappa_3) = 0.0825 + 0.2000i = 0.2163e^{i\arctan \frac{0.2000}{0.0825}}.$$

Obviously, The supporting degree of evidence for propositions  $\{\kappa_2\}$ ,  $\{\kappa_2, \kappa_3\}$   
100 and  $\{\kappa_1, \kappa_2, \kappa_3\}$  are 0.4989, 0.2848 and 0.2163 respectively. It is worth noting

that when  $\theta = 0$ , the above three CBPAs degenerate into BPAs, but the support degree of the proposition remains the original value. For a BPA, the degree of support for a proposition corresponds to the way the proposition is expressed. For a CBPA, the degree of support for a proposition corresponds to a set of expressions. This BPA is just one element of this set. This can be seen more clearly in Figure.1.

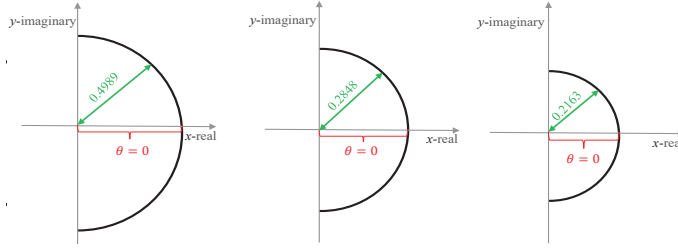


Figure 1: The relationship between BPA and CBPA

In complex evidence theory, there are some common functions, we define as follows:

**Definition 5.** (*Complex belief function*)

$$CBel(A) = \frac{\sum_{B \subseteq A} CM(B)}{\left| \sum_{C \subseteq \mathbb{K}} CM(C) \right|} \quad (10)$$

**Definition 6.** (*Complex plausibility function*)

$$CPl(A) = \frac{\sum_{B \cap A \neq \emptyset} CM(B)}{\left| \sum_{C \subseteq \mathbb{K}} CM(C) \right|} \quad (11)$$

$|CBel(A)|$  is the minimum support degree of evidence to proposition  $A$ , and  $|CPl(A)|$  is the maximum support degree of evidence to proposition  $A$ . Obviously,  $|CBel(A)| \leq |CPl(A)|$ , they form the upper and lower bounds of support degree  $[|CBel(A)|, |CPl(A)|]$ .

**Definition 7.** (*Complex Pignistic probability transformation*)

Pignistic probability transformation was proposed by Smet, which transformed

mass functions into probability distributions in the real number space[50]. Therefore, we also transformed complex mass function into probability distribution through the following equation:

$$BetP(A) = \sum_{B \subseteq \mathbb{K}} \frac{|A \cap B|}{|B|} \frac{|\mathbb{CM}(B)|}{1 - |\mathbb{CM}(\emptyset)|}, \forall A \subseteq \mathbb{K} \quad (12)$$

Let  $\mathbb{CM}_1$  and  $\mathbb{CM}_2$  be are two CBPA on  $\mathbb{K}$ ,  $A_i$  and  $B_j$  are subsets of  $\mathbb{K}$ . Thus, complex Dempster combination rule (CDCR) of  $\mathbb{CM}_1$  and  $\mathbb{CM}_2$  is denoted  $\mathbb{CM}_1 \otimes \mathbb{CM}_2$ , it is defined as follows:

**Definition 8.** (Complex Dempster combination rule)

$$\mathbb{CM}(A) = \begin{cases} \frac{\sum_{i,j:A_i \cap B_j = A} \mathbb{CM}_1(A_i) \mathbb{CM}_2(B_j)}{\sum_{i,j:A_i \cap B_j \neq \emptyset} |\mathbb{CM}_1(A_i) \mathbb{CM}_1(B_j)|} & A \neq \emptyset \\ 0 & A = \emptyset \end{cases} \quad (13)$$

In CDCR,  $K(\mathbb{CM}_1, \mathbb{CM}_2) = 1 - \sum_{i,j:A_i \cap B_j \neq \emptyset} |\mathbb{CM}_1(A_i) \mathbb{CM}_1(B_j)|$ , and  $K \in [0, 1]$ , it denotes the degree of conflict between evidence. The bigger the  $K$  value, the bigger the conflict degree, and the lower the credibility of the fusion results. In particular when  $K = 1$ , there is something wrong with the evidence that cannot be ignored. In addition, for  $\forall \mathbb{CM}_1$  and  $\mathbb{CM}_2$ ,  $K(\mathbb{CM}_1, \mathbb{CM}_2)$  has some properties as follows:

**P(1):Nonnegativity**,  $K(\mathbb{CM}_1, \mathbb{CM}_2) \leq 0$ ;

**P(2):Symmetry**,  $K(\mathbb{CM}_1, \mathbb{CM}_2) = K(\mathbb{CM}_2, \mathbb{CM}_1)$ ;

**P(3):Bounded.**  $0 \leq K(\mathbb{CM}_1, \mathbb{CM}_2) \leq 1$ .

**Example 3.2** Supposing that there are two CBPAs  $\mathbb{CM}_1$  and  $\mathbb{CM}_2$  in  $\mathbb{K} = \{\kappa_1, \kappa_2\}$ , we express  $\mathbb{CM}_1$  and  $\mathbb{CM}_2$  as follows:

$$\mathbb{CM}_1(\kappa_1) = \rho e^{i\theta}, \quad \mathbb{CM}_1(\kappa_2) = (1 - \rho) e^{i(\frac{\pi}{2} - \theta)}.$$

$$\mathbb{CM}_2(\kappa_1) = (1 - \rho) e^{i(\frac{\pi}{2} - \theta)}, \quad \mathbb{CM}_1(\kappa_2) = \rho e^{i\theta}.$$

where  $\rho \in [0, 1]$ ,  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ . Then, the calculation results of conflict coefficient  $K$  are shown in Figure 2. As can be seen from the Figure.2, when  $\rho = 0$  or  $\rho = 1$ , for any  $\theta$ ,  $K$  is always going to be 1. When  $\rho = 0.5$  and  $\theta = 0.1492$ , then  $K = 0.5$ .

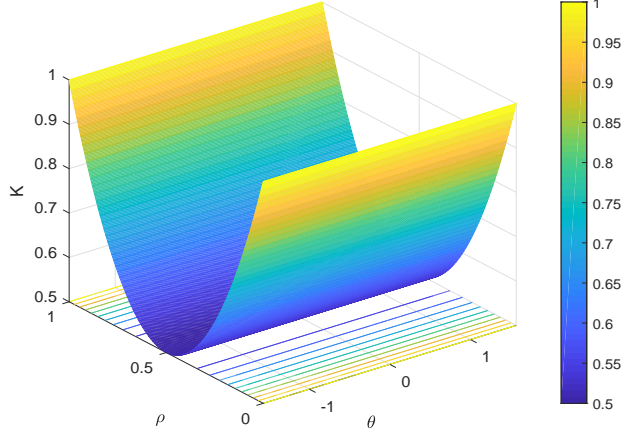


Figure 2: The conflict coefficient  $K$  in Example 3.2

**Example 3.3** Exist two CBPAs  $\mathbb{CM}_1$  and  $\mathbb{CM}_2$  in  $\mathbb{K} = \{\kappa_1, \kappa_2, \kappa_3\}$  are  
 145 described as follows:

$$\begin{aligned}\mathbb{CM}_1: \mathbb{CM}_1(\kappa_2) &= 1; \\ \mathbb{CM}_2: \mathbb{CM}_2(\kappa_2) &= 0.4921 + 0.0739i, \\ \mathbb{CM}_2(\kappa_2, \kappa_3) &= 0.2937 + 0.0982i, \\ \mathbb{CM}_2(\kappa_1, \kappa_2, \kappa_3) &= 0.1344 + 0.1381i.\end{aligned}$$

150 In this example, the  $K$  is 0.0289, if we change  $\mathbb{CM}_2$  to  $\mathbb{CM}_2(\kappa_2) = 1$ , under this condition, the calculation result of  $K$  is 0, if by Dempster-Shafer evidence theory analysis, the  $K$  is 0 in both cases. Obviously, the conflict coefficient  $K$  in the complex evidence theory has a stricter requirement on the expression form of CBPAs.

155 **Example 3.4** Supposing there are four CBPA on the frame of discernment  $\mathbb{K} = \{\kappa_1, \kappa_2, \kappa_3\}$ , they are described as follows:

$$\begin{aligned}\mathbb{CM}_1: \mathbb{CM}_1(\kappa_1) &= 0.7765 + 0.0009i, \\ \mathbb{CM}_1(\kappa_1, \kappa_2) &= 0.0756 + 0.0674i, \\ \mathbb{CM}_1(\kappa_1, \kappa_2, \kappa_3) &= 0.0464 + 0.1130i; \\ 160 \mathbb{CM}_2: \mathbb{CM}_2(\kappa_2) &= 0.5900 - 0.0068i, \\ \mathbb{CM}_2(\kappa_2, \kappa_3) &= 0.4091 + 0.0263i\end{aligned}$$

$$\begin{aligned}\mathbb{CM}_3: \mathbb{CM}_3(\kappa_2) &= 0.5430 - 0.0177i, \\ \mathbb{CM}_3(\kappa_2, \kappa_3) &= 0.4564 + 0.0166i; \\ \mathbb{CM}_4: \mathbb{CM}_4(\kappa_2) &= 0.5580 + 0.0287i, \\ \mathbb{CM}_4(\kappa_2, \kappa_3) &= 0.3425 + 0.0649i, \\ \mathbb{CM}_4(\kappa_1, \kappa_2, \kappa_3) &= 0.0910 + 0.0177i.\end{aligned}$$

Then,  $\mathbb{CM}_1 \oplus \mathbb{CM}_2 \oplus \mathbb{CM}_3 \oplus \mathbb{CM}_4$  is calculated as follows:

$$\begin{aligned}\mathbb{CM}(\kappa_2) &= 0.4417 + 0.8452i, \\ \mathbb{CM}(\kappa_2, \kappa_3) &= 0.0047 + 0.0461i.\end{aligned}$$

In the combined results,  $|\mathbb{CM}(\kappa_2)| + |\mathbb{CM}(\kappa_2, \kappa_3)| = 1$ . In addition, the support degree of  $\kappa_2$  is the biggest,  $|\mathbb{CM}(\kappa_2)| = 0.9537$ , it is a non-counterintuitive result.

#### 4. The generation method and application of complex mass function

In this section, we propose the generation method of complex mass function. In addition, we apply the generated complex mass function to the target recognition of the iris dataset.

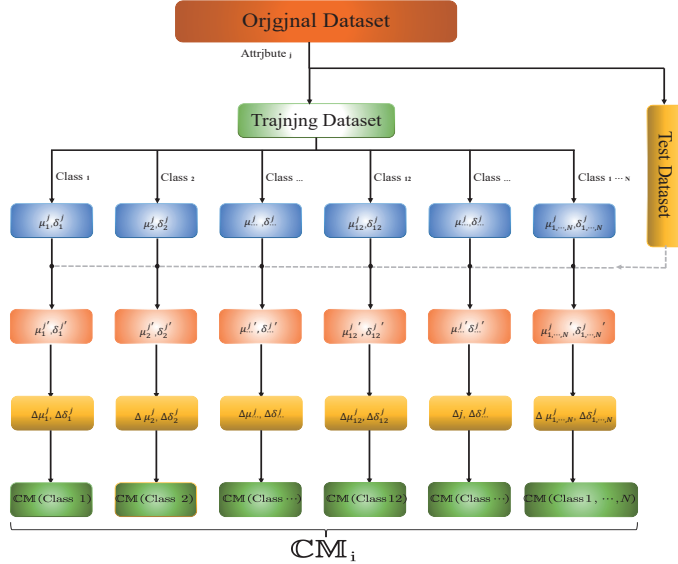


Figure 3: The generation method of Complex mass function

Methods Described: Supposing there are  $N$  targets, denoted as  $\{Class1, Class2, \dots, Classj, \dots, ClassN\}$ , each target has  $K$  attributes, represented as  $\{Attribute1, Attribute2, \dots, Attributei, \dots, AttributeK\}$ .

Thus, we can use the process shown in Figure.3 to generate the complex mass

180 functions for the  $j$ th attribute.

- Step 1: Select the training set and test set from the original data set.
- Step 2:  $N$  class correspond to  $2^N - 1$  propositions, denoted as  $\mathbb{N} = \{Calss_1, Calss_2, \dots, Calss_{12}, \dots, Calss_{1, \dots, N}\}$ , where  $Calss_{12}$  is the union of  $Calss_1$  and  $Calss_2$ . Calculating the mean  $\mu_p^j$  and standard deviation  $\delta_p^j$  of  $p$ th proposition.
- Step 3: Select an unknown target from the test set and add it to the training set, and calculate the mean  $\mu_p^{j'}$  and standard deviation  $\delta_p^{j'}$  of the  $p$ th proposition again.
- Step 4: The complex mass function of the  $p$ th proposition is generated by

190 the following equation:

$$\mathbb{CM}_j(Class_p) = \left( \frac{1}{e^{|\delta_p^{j'} - \delta_p^j|}} \right) e^{i(\mu_p^{j'} - \mu_p^j)} = \left( \frac{1}{e^{|\Delta \delta_p^j|}} \right) e^{i(\Delta \mu_p^j)} \quad (14)$$

Experimental verification: we selected Iris data set, Seed data set and Banknote data set, Blood Transfusion Service Center data set (BTSC) and Fertility\_Diagnosis data set from UCI database (<http://archive.ics.uci.edu/ml/index.php>) for experiments. The details are shown in Table 1. 60% of the data from each kind

195 of data set is randomly selected as the training set and the rest as the test set. We conducted 100 experiments, and the results are shown in Figure.4. As can be seen from Figure.4, compared with the mass function in the real number plane[51], the description of uncertain information by complex mass function can improve the accuracy of target recognition, which indicates that the description

200 of uncertain information by complex mass function is more reasonable.

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**Algorithm 1** :Generation of complex mass functions

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**Input:** *Training set and test set* /\* They all have n classes, each class has k attributes. \*/

**Output:** *Complex Mass Functions for the test set*

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1: for  $j = 1 \rightarrow k$  do
2:   for  $p = 1 \rightarrow 2^N - 1$  do
3:     Calculating  $\delta_p^j, \mu_p^j, \delta_p^{j'}, \mu_p^{j'}$ .
4:      $\Delta\delta_p^j = \delta_p^{j'} - \delta_p^j, \Delta\mu_p^j = \mu_p^{j'} - \mu_p^j$ 
5:      $\mathbb{CM}_j(Class_p) = \left( \frac{1}{e^{|\Delta\delta_p^j|}} \right) e^{i(\Delta\mu_p^j)}$ 
6:   end for
7: return  $\mathbb{CM}_j$ 
8: end for
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Table 1: The details of the datasets

Dataset	Instance	Class	Attribute	AttrDist	Missing value
Iris	150	3	4	4R	No
Seeds	210	3	7	7R	No
Banknote	1372	2	4	4R	No
BTSC	748	2	4	4R	No
Fertility_Diagnosis	100	2	9	4R5I	No

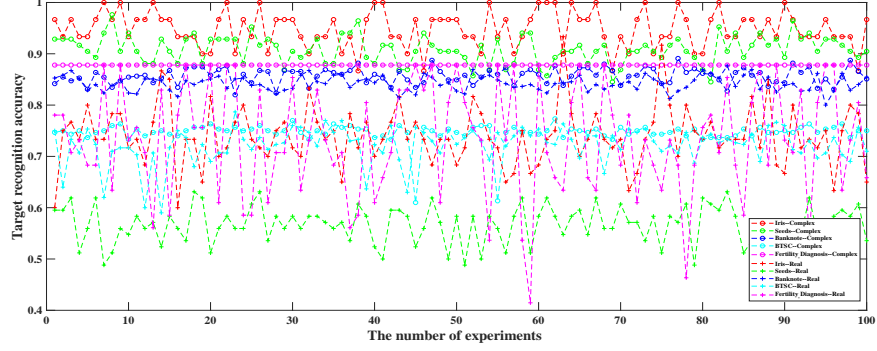


Figure 4: The target recognition accuracy based on mass functions in real plane and complex plane

## 5. Conclusion

In order to describe the uncertain information of multi-dimensional features, we extend the Dempster-Shafer evidence theory from the real number space to the complex number space, and propose the complex Dempster-Shafer evidence theory. Under the framework of complex Dempster-Shafer evidence theory, we define complex mass function, complex belief function and etc. Meanwhile, we also propose the complex Dempster combinatorial rule, which satisfies the commutative law of multiplication. When complex mass function degenerates into traditional mass function, complex Dempster combination rule will also degenerate into Dempster combination rule in real space. Then, we proposed a method to generate complex mass function based on the mean and variance changes of sample data. The results show that compared with the mass function in the real number space, the complex mass function can effectively improve the target recognition rate under the framework of complex evidence theory. In subsequent work, we will explain complex Dempster-shafer evidence theory from an algebraic perspective, which will help to perfect the theoretical basis of complex Dempster-shafer evidence theory, and we will also apply complex Dempster-shafer evidence theory to target classification, cluster analysis, intelligent decision making and other fields to deal with more practical problems. In addition,

220 complex probability is related to the description of quantum system[52, 53], so  
we will use complex evidence theory to further solve some problems of quantum  
system, such as entanglement effect, interference effect and etc.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or  
225 personal relationships that could have appeared to influence the work reported  
in this paper.

### Acknowledgment

The work is partially supported by National Natural Science Foundation of  
China (Grant Nos. 61973332), JSPS Invitational Fellowships for Research in  
230 Japan (Short-term).

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